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Oscillations of the system of linear nonautonomous difference equations

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Abstract

We consider the linear nonautonomous system of difference equations $x_{n+1} - x_n + P(n)x_{n-k} = 0$, $n = 0, 1, 2, \ldots,$ where $k \in Z$, $P(n) \in R^{rxr}$. We obtain sufficient conditions for the system to be oscillatory. The conditions based on the eigenvalues of the matrix coefficients of the system.

Keywords: Difference; Oscillation; Nonautonomous

1. Introduction

Recently there has been a lot of studying concerning the oscillation of all solutions of the delay difference equations

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $\{p_n\}$ is sequence of nonnegative real numbers and k is a positive integer.

By a solution of Eq. (1) we mean a sequence $\{x_n\}$ which is defined for $n \geqslant -k$ and which satisfies Eq. (1) for $n \geqslant 0$. A solution $\{x_n\}$ of Eq. (1) is said to be oscillatory if the terms x_n of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory.

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In 1989, Erbe and Zhang [1] proved that every solution of Eq. (1) oscillates if

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}} \tag{2}$$

or

$$\limsup_{n\to\infty} \sum_{s=n-k}^{n} p_s > 1 \tag{3}$$

be satisfied.

In the same year, Ladas et al. [2] proved that every solution of Eq. (1) oscillatory if

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} p_i\right] > \frac{k^k}{(k+1)^{k+1}} \tag{4}$$

holds.

When $p(n) = p$ for $n = 0, 1, 2, \ldots$, Eq. (1), conditions (2) and (4) reduce to

$$p > \frac{k^k}{(k+1)^{k+1}} \tag{5}$$

which is a necessary and sufficient condition to be oscillatory of all solutions of the difference equation

$$x_{n+1} - x_n + p x_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $p \in (0, \infty)$, $k \in Z^+$.

In 1990, Chuanxi et al. [3] have given the necessary and sufficient conditions to be oscillatory of all solutions of the linear, autonomous difference equation

$$x_{n+1} - x_n + P x_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \tag{7}$$

where $P \in R^{rxr}$, $k \in Z$, and $\det P \neq 0$. By a solution of system (7) we mean a sequence $\{x_n\}$ of vectors in R^r for $n = 0, 1, 2, \ldots$, which satisfies system (7). Let $\{x_n\}$ be a solution of system (7) with $x_n = [x_n^1, x_n^2, \ldots, x_n^r]$ for $n = 0, 1, 2, \ldots$. We say that the solution $\{x_n\}$ oscillates componentwise or simply oscillates if each component $\{x_n^i\}$ for $i = 1, 2, \ldots, r$ oscillates. Otherwise the solution is called nonoscillatory.

Lemma 1.1. *Let $k \in Z$, $P \in R^{rxr}$, and* $\det P \neq 0$. *Then every solution of system* (7) *oscillates if and only if*

$$\det\left[(\lambda - 1)I + P\lambda^{-k}\right] = 0$$

the characteristic equation of system (7) *has no positive roots* [4].

2. Sufficient conditions for oscillations

The main result in this section is the following theorem involving sufficient conditions for the oscillation of all solutions of system

$$y_{n+1} - y_n + P(n) y_{n-k} = 0, \quad n = 0, 1, 2, \ldots. \tag{8}$$

Theorem 2.1 (The main theorem). *Let* $P(n) \in R^{rxr}$, $k \in Z$, *and* $\det P(n) \neq 0$ *for* $\forall n \in N$. *Then every solution of system* (8) *oscillates if one of the following conditions holds*:

(i) $k = 0$ *and* $\limsup_{n\to\infty} \mu(I - P(n)) < 0$;
(ii) $k = -1$ *and* $\limsup_{n\to\infty} \mu(I + P(n)) < 0$;
(iii) $k \geqslant 1$ *and* $\liminf_{n\to\infty} \mu(P(n)) > k^k/(k+1)^{k+1}$;
(iv) $k \leqslant -2$ *and* $\limsup_{n\to\infty} \mu(P(n)) < k^k/(k+1)^{k+1}$,

where $\mu(P(n))$ *denotes any real eigenvalue of* $P(n)$.

Proof. (i) $k = 0$. In this case the system (8) becomes

$$y_{n+1} - y_n + P(n)y_n = 0. \tag{9}$$

Characteristic equation of system (9) is

$$\det\big[\lambda I - I + P(n)\big] = 0 \tag{10}$$

which can also be written as

$$\det\big[\lambda I - (I - P(n)\big] = 0.$$

Thus Eq. (10) eventually has no positive roots if $(I - P(n))$ eventually has no positive eigenvalues. That is

$$\limsup_{n\to\infty} \mu\big(I - P(n)\big) < 0.$$

(ii) $k = -1$. In this case the system (8) becomes

$$y_{n+1} - y_n + P(n)y_{n+1} = 0. \tag{11}$$

Characteristic equation of system (11) is

$$\det\big[\lambda I - I + P(n)\lambda\big] = 0 \tag{12}$$

and

$$\det\big[I - \lambda^{-1} I + P(n)\big] = 0$$

which can also be written as

$$\det\big[\lambda^{-1} I - \big(I + P(n)\big)\big] = 0.$$

Thus Eq. (12) eventually has no positive roots if $(I + P(n))$ eventually has no positive eigenvalues. That is

$$\limsup_{n\to\infty} \mu\big(I + P(n)\big) < 0.$$

(iii) $k \geqslant 1$. In this case the system (8) characteristic equation becomes

$$\det\big[\lambda I - I + P(n)\lambda^{-k}\big] = 0 \tag{13}$$

and

$$\det\big[\lambda^{k+1} I - \lambda^k I + P(n)\big] = 0$$

which can also be written as

$$\det\big[\big(\lambda^k-\lambda^{k+1}\big)I-P(n)\big]=0.$$

In this case we take the following function $v(\lambda)$

$$v(\lambda)=\lambda^k-\lambda^{k+1}.$$

It is clear that $v(\lambda)$ is a continuous function on $(0,\infty)$,

$$\max_{\lambda>0} v(\lambda)=\frac{k^k}{(k+1)^{k+1}} \quad \text{and} \quad \lim_{\lambda\to\infty} v(\lambda)=-\infty.$$

Thus the image of $(0,\infty)$ under $v(\lambda)$ is $(-\infty,k^k/(k+1)^{k+1}]$. Therefore Eq. (13) eventually has no positive roots on the interval $(-\infty,k^k/(k+1)^{k+1}]$ if $P(n)$ eventually has no eigenvalues in $(-\infty,k^k/(k+1)^{k+1}]$. That is

$$\liminf_{n\to\infty}\mu\big(P(n)\big)>\frac{k^k}{(k+1)^{k+1}}.$$

(iv) $k\leqslant -2$. In this case the system (8) characteristic equation becomes

$$\det\big[\lambda I-I+P(n)\lambda^{-k}\big]=0$$

and can be taken as $v(\lambda)=\lambda^k-\lambda^{k+1}$

$$\min_{\lambda>0} v(\lambda)=\frac{k^k}{(k+1)^{k+1}} \quad \text{and} \quad \lim_{\lambda\to 0} v(\lambda)=\infty.$$

Thus the image of $(0,\infty)$ under $v(\lambda)$ is $[k^k/(k+1)^{k+1},\infty)$. Therefore Eq. (13) eventually has no positive roots on the interval $[k^k/(k+1)^{k+1},\infty)$ if $P(n)$ eventually has no eigenvalues in $[k^k/(k+1)^{k+1},\infty)$. That is

$$\limsup_{n\to\infty}\mu\big(P(n)\big)<\frac{k^k}{(k+1)^{k+1}}. \qquad \Box$$

Corollary 2.2. *Let $P(n)\in R^{rxr}$, $k\in Z$, and* $\det P(n)\neq 0$ *for $\forall n\in N$. Then every solution of system* (8) *oscillates if and only if one of the following conditions holds*:

(i) $k=0$ *and* $\mu(I-P(n))<0$ *for* $\forall n\in N$;
(ii) $k=-1$ *and* $\mu(I+P(n))<0$ *for* $\forall n\in N$;
(iii) $k\geqslant 1$ *and* $\mu(P(n))>k^k/(k+1)^{k+1}$ *for* $\forall n\in N$;
(iv) $k\leqslant -2$ *and* $\mu(P(n))<k^k/(k+1)^{k+1}$ *for* $\forall n\in N$,

where $\mu(P(n))$ denotes any real eigenvalue of $P(n)$.

Proof. For the proof see Lemma 1.1 and Theorem 2.1. $\Box$

Theorem 2.3. *Suppose that $\{p_n\}$ is a nonnegative sequence of real numbers and let k be a positive integer. Then the following statements are equivalent*:

(a) $\liminf_{n\to\infty} p_n > \dfrac{k^k}{(k+1)^{k+1}}$;

(b) $\liminf_{n\to\infty} \left[\dfrac{1}{k}\sum_{i=n-k}^{n-1} p_i\right] > \dfrac{k^k}{(k+1)^{k+1}}$.

Proof. (a) $\Rightarrow$ (b). If (a) holds, that is to say

$$\liminf_{n\to\infty} p_n = \liminf_{n\to\infty} p_{n-1} = \cdots = \liminf_{n\to\infty} p_{n-k}.$$

Then we can write the expression (a) as

$$\liminf_{n\to\infty} p_n = \frac{1}{k}\Big[\liminf_{n\to\infty} p_{n-k} + \cdots + \liminf_{n\to\infty} p_{n-1}\Big] > \frac{k^k}{(k+1)^{k+1}}$$

so

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} p_i\right] > \frac{k^k}{(k+1)^{k+1}}.$$

(b) $\Rightarrow$ (a). If (b) holds, then

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1} p_i\right] > \frac{k^k}{(k+1)^{k+1}}$$

and

$$\left[\frac{1}{k}\Big(\liminf_{n\to\infty} p_{n-k} + \cdots + \liminf_{n\to\infty} p_{n-1}\Big)\right] > \frac{k^k}{(k+1)^{k+1}}$$

finally

$$\left[\frac{1}{k}\Big(k\liminf_{n\to\infty} p_n\Big)\right] > \frac{k^k}{(k+1)^{k+1}}. \qquad \square$$

Remark 2.1. In [1,2] it has been proved respectively that the conditions (a) and (b) in the above theorem are the sufficient conditions for the oscillations of all solutions of Eq. (1).

Theorem 2.4. *Suppose that* $\{\mu(P(n))\}$ *is a sequence of real numbers and let* k *be a positive integer. Then the following statements are equivalent. Where* $\mu(P(n))$ *defined as in Theorem* 2.1:

(a) $\liminf_{n\to\infty} \mu\big(P(n)\big) > \dfrac{k^k}{(k+1)^{k+1}}$;

(b) $\liminf_{n\to\infty} \left[\dfrac{1}{k}\sum_{i=n-k}^{n-1} \mu\big(P(i)\big)\right] > \dfrac{k^k}{(k+1)^{k+1}}$.

Proof. (a) $\Rightarrow$ (b). If (a) holds, that is to say

$$\liminf_{n\to\infty} \mu\big(P(n)\big) = \cdots = \liminf_{n\to\infty} \mu\big(P(n-k)\big).$$

Then we can write

$$\liminf_{n\to\infty}\mu\bigl(P(n)\bigr)=\frac{1}{k}\Bigl[\liminf_{n\to\infty}\mu\bigl(P(n-k)\bigr)+\cdots+\liminf_{n\to\infty}\mu\bigl(P(n-1)\bigr)\Bigr]$$
$$>\frac{k^k}{(k+1)^{k+1}}$$

and

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1}\mu\bigl(P(i)\bigr)\right]>\frac{k^k}{(k+1)^{k+1}}.$$

(b) $\Rightarrow$ (a). If (b) holds, then

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1}\mu\bigl(P(i)\bigr)\right]>\frac{k^k}{(k+1)^{k+1}}$$

and

$$\left[\frac{1}{k}\Bigl(\liminf_{n\to\infty}\mu\bigl(P(n-k)\bigr)+\cdots+\liminf_{n\to\infty}\mu\bigl(P(n-1)\bigr)\Bigr)\right]>\frac{k^k}{(k+1)^{k+1}},$$

finally

$$=\left[\frac{1}{k}\Bigl(k\liminf_{n\to\infty}\mu\bigl(P(n)\bigr)\Bigr)\right]>\frac{k^k}{(k+1)^{k+1}}$$

and the proof is completed. □

Corollary 2.5. *Suppose that* $\{\mu(P(n))\}$ *is a sequence of real numbers and let* k *be a positive integer. Then the condition* (b) *of Theorem* 2.4 *that is*

$$\liminf_{n\to\infty}\left[\frac{1}{k}\sum_{i=n-k}^{n-1}\mu\bigl(P(i)\bigr)\right]>\frac{k^k}{(k+1)^{k+1}}$$

is a sufficient condition for every solution of system (8) *to be oscillatory.*

Proof. For the proof see Theorems 2.1 and 2.4. □

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